Factoring Methods

When you are trying to factor a polynomial, there are three general steps you want to follow:
1. See if there is a Greatest Common Factor
2. See if you can Factor by Grouping
3. See if it is a Special Case

Common Factor or Greatest Common Factor (GCF)

A **Common Factor** is something you can divide each term of a polynomial by evenly. The **Greatest Common Factor** (GCF) is the largest common factor you can evenly divide each term of a polynomial by. When you are factoring, you always want to remove the GCF. Remember that 1 is not a GCF. An easy way to see what is common to all the terms is to individually factor each term. After factoring each term in Example 1, it easy to see that **2** is the largest number that is common to all the terms in the polynomial and therefore the GCF. The next step is to put the GCF on the outside of the parenthesis and put what is left inside the parenthesis.

Example 1: $2x + 2$
                $\textbf{2} * x + \textbf{2} * 1$
                $\textbf{2}(x + 1)$

In Example 2, the same process is done, but in this case it is not a number that is common to both terms. It is the variable $\boldsymbol{x}$ that is common, so $\boldsymbol{x}$ can be moved to the outside of the parenthesis and $3x + 4$ can be left inside the parenthesis.

Example 2: $3x^2 + 4x$
                $3 * x * \boldsymbol{x} + 4 * \boldsymbol{x}$
                $\boldsymbol{x}(3x + 4)$

In Example 3, we again follow the same process. First, each term is factored individually. Then we determine what is common in all the terms. In this case, it is $\boldsymbol{5x^2}$, so $\boldsymbol{5x^2}$ will be put on the outside of the parenthesis and $x^2 + 7x + 2$ will be put inside the parenthesis.

Example 3:
$5x^4 - 35x^3 + 10x^2$
$5x^4 = \boldsymbol{5} * \boldsymbol{x} * \boldsymbol{x} * x * x$ $35x^3 = 7 * \boldsymbol{5} * \boldsymbol{x} * \boldsymbol{x} * x$ $10x^2 = \boldsymbol{5} * 2 * \boldsymbol{x} * \boldsymbol{x}$
GCF = $\boldsymbol{5x^2}$
Factored form: $\boldsymbol{5x^2}(x^2 - 7x + 2)$

Factoring by Grouping

Before Factoring by Grouping, remember to always see if there is a common factor. If there is one, factor out the GCF *before* trying to factor by grouping.

Example 4:   10abx + 15bx - 8ax - 12x
                    x(10ab + 15b - 8a - 12)
                    x[(10ab + 15b) + (-8a - 12)]
                    x[5b(2a + 3) + -4(2a + 3)]
                    x(2a + 3)(5b - 4)

Factoring by grouping requires that you have four terms. What you want to do is group the terms into two groups of two terms each. When deciding which terms to put together, think about which terms share a common factor. In Example 5, it does not matter which terms are grouped together because all the terms share a common factor other than one. However, in Example 6A, you cannot group 6xy and -5 together and -15x and 2y together because these two groups do not share a common factor other than one. At least one group must share a common factor other than one. Also note that in all the examples, a plus sign was put between the parentheses. It is important that there be a plus sign between them. The next step is to factor out the GCF in each set of parentheses. See above for how to do this. After factoring out the GCF, the insides of the parentheses should be the same. If the groups are not the same as in Example 6A and 7, then either you need to change the grouping to make it work as in Example 6 or the problem is not factorable by grouping as in Example 7. If the insides of the parentheses match, then write what is in the parentheses one time and then write what is outside the parentheses in another set of parentheses.

Example 5:   2ab + 12a + 3b + 18
                    (2ab + 12) + (3b + 18)
                    2a(b + 6) + 3(b + 6)
                    (b + 6)(2a + 3)

Example 6A: 6xy - 5 - 15x + 2y   6B: 6xy - 5 - 15x + 2y
                    (6xy - 5) + (-15x +2y)   (6xy - 15x) + (-5 + 2y)
                    1(6xy - 5) + (-15x + 2y)   3x(2y - 5) + 1(-5 + 2y)
                    The parentheses don't match.   (2y - 5)(3x + 1)

Example 7:   x + 2xy + 3y + 5
                    (x + 2xy) + (3y + 5)
                    x(1 + 2y) + (3y + 5)
                    The parentheses don't match, and the problem can't be rearranged to work, so it is not factorable by grouping.

Factoring by Grouping is also used to factor problems in the form $ax^2 + bx + c$. The letters *a*, *b*, and *c* represent numbers. If you do not see a number in front of one of the x terms, then the number is 1.

Example 8: $4x^2 + x - 34$
$a = 4$
$b = 1$
$c = -34$

The way to factor a trinomial, a polynomial with three terms, is to change it from three terms to four. The first step is to find two numbers that add together to get *b* and multiply together to get *ac*. An easy way to find the numbers is to list the factors of *ac*. If you cannot find two numbers that will do this, then the problem is not factorable by grouping. Once you find the two numbers, break the *b* into those numbers. After that, the problems are just like Examples 4-7.

Example 9: $x^2 - 5x + 6$ __ + __ = -5 and __ * __ = 6 * 1 = 6
$x^2 -3x -2x +6$ Factors of 6 :
$(x^2 - 3x) + (-2x + 6)$
$x(x - 3) + -2(x - 3)$
$(x -3)(x - 2)$

1	6
-1	-6
2	3
-2	-3

$-2 + -3 = -5$
$-2 * -3 = 6$

Example 10: $2x^2 + 13x + 15$ __ + __ = 13 and __ * __ = 2 * 15 = 30
$2x^2 + 3x + 10x + 15$ Factors of 30:
$(2x^2 + 3x) + (10x + 15)$
$x(2x + 3) + 5(2x + 3)$
$(2x + 3)(x + 5)$

1	30
-1	-30
2	15
-2	-15
3	10
-3	-10
5	6
-5	-6

$3 + 10 = 13$
$3 * 10 = 30$

Example 11: $x^2 + 4x - 9$ __ + __ = 4 and __ * __ = 1 * 9 = 9
Factors of 9:

No factor of 9 can add to 4, so the trinomial is not factorable over the integers.

1	9
-1	-9
3	3
-3	-3

Special Case Factoring

There are some special cases for factoring that you just have to memorize.

- $a^2 - b^2 = (a + b)(a - b)$ Difference of Perfect Squares
- $a^3 + b^3 = (a + b)(a^2 - ab + b^2)$ Sum of Cubes
- $a^3 - b^3 = (a - b)(a^2 + ab + b^2)$ Difference of Cubes

You need to be able to recognize a special case and be able to follow the rule.

Example 12: $x^2 - 4$ Case: $a^2 - b^2 = (a + b)(a - b)$
Both x^2 and 4 are perfect squares.
$x^2 = x * x$ and $4 = 2^2 = 2 * 2$

Answer: $(x + 2)(x - 2)$

Example 13: $x^3 + 27$ Case: $a^3 + b^3 = (a + b)(a^2 - ab + b^2)$
Both x^3 and 27 are perfect cubes.
$x^3 = x * x * x$ and $27 = 3^3 = 3 * 3 * 3$

Answer: $(x + 3)(x^2 - 3x + 3^2)$
$(x + 3)(x^2 - 3x + 9)$

Example 14: $8x^3 - 125$ Case: $a^3 - b^3 = (a - b)(a^2 + ab + b^2)$
Both $8x^3$ and 125 are perfect cubes.
$8x^3 = 2 * 2 * 2 * x * x * x$ and $125 = 5^3 = 5 * 5 * 5$

Answer: $(2x - 5)[(2x)^2 + 10x + (5)^2]$
$(2x - 5)(4x^2 + 10x + 25)$

Substitution

Substitution is used to factor trinomials that are in the form: $ax^{2m} + bx^m + c$. Again, the letters *a*, *b*, and *c* represent numbers, and *m* also represents a number. What you do is substitute *u* for x^2 in the problem, then you factor normally. After you have your answer, you then substitute x^2 back into the problem. Example 15 shows how to use substitution in a problem.

Example 15: $x^4 - 4x^2 - 45$ $u = x^2$
$u^2 - 4u - 45$ __ + __ = -4   and __ * __ = -45
$u^2 - 9u + 5u - 45$
$u(u - 9) + 5(u - 9)$
$(u + 5)(u - 9)$
$(x^2 + 5)(x^2 - 9)$
$(x^2 + 5)(x + 3)(x - 3)$

-1	45
1	-45
-3	15
3	-15
-5	9
5	-9

5 - 9 = -4 and (5)(-9) = -45

Example 16: $x^4 - 81$ $u = x^2$
$u^2 - 81$
$(u + 9)(u - 9)$
$(x^2 + 9)(x^2 - 9)$ Difference of Perfect Squares on $(x^4 - 81)$
$(x^2 + 9)(x + 3)(x - 3)$ Difference of Perfects Squares on $(x^2 - 9)$

Example 17: $(x - 2)^2 - 9$ $u = x - 2$
$u^2 - 9$
$(u + 9)(u - 9)$
$(x - 2 + 9)(x - 2 - 9)$
$(x + 7)(x - 11)$

Quick Steps to Factoring

1. Common Factor   (Examples 1 - 3)
    a.   If there is a common factor in the problem, factor it out.

2. Factor by Grouping
    a.   There are three terms   (Examples 9 - 11)
        i.   Find two numbers that ___ + ___ = b   and ___ * ___ = ac
        ii. Spit the b term into the two numbers.
        iii. Follow the steps in 2b.
    b.   There are four terms   (Examples 5 - 7)
        i.   Group the terms into two sets of parentheses.
        ii. Factor the GCF out of each parenthesis.
        iii. If the parentheses are the same after removing the GCF, then write the parentheses once and write what you factored out of the parentheses in a second set of parenthesis.

3. Special Factoring   (Examples 12 - 14)
    a.   Is it a special case of factoring? If so follow the appropriate rule.
        i. $(a^2 + b^2) = (a + b)(a - b)$
        ii. $a^3 + b^3 = (a + b)(a^2 - ab + b^2)$
        iii. $a^3 - b^3 = (a - b)(a^2 + ab + b^2)$

Sample Problems

1. $2x^2 - 18$

2. $3y^2 - 48$

3. $a^4 - 16$

4. $5a^2 - 30a + 45$

5. $4a^2 + 16a + 16$

6. $-x^2 + 50x - 625$

7. $ax - bx + ay - by$

8. $2ax + 3 + x + 6a$

9. $m^3 + n^3$

10. $r^3 - s^3$

11. $64x^3 - 1$

12. $8x^3 + 1$

13. $27y^3 - 1$

14. $125y^3 - 1$

15. $3a^2 - 2ax - 3a + 2x$

16. $a^2 - 2a + ab - 2b$

17. $125 - y^3$

18. $x^6 - 27$

19. $x^6 + 125$

20. $m^3n^6 + d^3$

21. $a^3 - a^2b - a + b$

22. $x^2 + 6x + 5$

23. $x^2 - 4x + 3$

24. $n^2 + 5n + 6$

25. $n^2 - 10n + 25$

26. $m^2 + 3ms - 4s^2$

27. $y^2 + 4y - 12$

28. $y^2 - y - 30$

29. $t^2 - 14t - 72$

30. $6 - x - x^2$

31. $36 + 5x - x^2$

32. $36s^2 + 12s + 1$

33. $6x^2 + 30x - 900$

34. $2a^4 - 10a^3 - 72a^2$

35. $2x^3 - 3x^2 - 2x + 3$

36. $(x - 1)^2 - 4$

37. $(x + 2)^2 - (y - 3)^2$

38. $16 - (2x - 1)^2$

39. $4a^2 - 4ab - 36 + b^2$

40. $2a^3 - 16a^2 + 32a$

Answers:

1. $2(x - 3)(x + 3)$

2. $3(y - 4)(y + 4)$

3. $(a - 2)(a + 2)(a^2 + 4)$

4. $5(a - 3)^2$

5. $4(a + 2)^2$

6. $-1(x - 25)^2$

7. $(x + y)(a - b)$

8. $(x + 3)(2a + 1)$

9. $(m + n)(m^2 - mn + n^2)$

10. $(r - s)(r^2 + rs + s^2)$

11. $(4x - 1)(16x^2 + 4x + 1)$

12. $(2x + 1)(4x^2 - 2x + 1)$

13. $(3y - 1)(9y^2 + 3y + 1)$

14. $(5y - 1)(25y^2 + 5y + 1)$

15. $(a - 1)(3a - 2x)$

16. $(a - 2)(a + b)$

17. $(5 - y)(25 + 5y + y^2)$

18. $(x^2 - 3)(x^4 + 3x^2 + 9)$

19. $(x^2 + 5)(x^4 - 5x^2 + 25)$

20. $(mn^2 + d)(m^2n^4 - mn^2d + d^2)$

21. $(a - 1)(a + 1)(a - b)$

22. $(x + 5)(x + 1)$

23. $(x - 3)(x - 1)$

24. $(n + 2)(n + 3)$

25. $(n - 5)^2$

26. $(m - s)(m + 4s)$

27. $(y + 6)(y - 2)$

28. $(y - 6)(y + 5)$

29. $(t - 18)(t + 4)$

30. $(3 + x)(2 - x)$

31. $(9 - x)(4 + x)$

32. $(6s + 1)^2$

33. $6(x + 15)(x - 10)$

34. $2a^2(a - 9)(a + 4)$

35. $(x - 1)(x + 1)(2x - 3)$

36. $(x + 1)(x - 3)$

37. $(x - y + 5)(x + y - 1)$

38. $(5 - 2x)(3 + 2x)$

39. $4a^2 - 4ab + b^2 - 36$
    $(2a - b)^2 - 36$
    $(2a - b + 6)(2a - b - 6)$

40. $2a(a - 4)^2$